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Kochen–Specker Theorem in the Modal Interpretation of Quantum Mechanics

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According to the modal interpretation of quantum mechanics, subsystems of a quantum mechanical system have definite properties, the set of definite properties forming a partial Boolean algebra. It is shown that these partial Boolean algebras have no common extension (as a partial Boolean subalgebra of the properties of the total system) that is embeddable in a Boolean algebra. One has thus either to restrict the rules to preferred subsystems (Healey), or to advocate a shift in metaphysics (Dieks).

1. INTRODUCTION

The riddles in the foundations of quantum mechanics are closely linked with the question of *realism* in the interpretation of quantum mechanics. An instrumentalist interpretation of quantum mechanics has no measurement problem, because it only seeks to describe regularities arising on the level of macroscopic preparation and registration procedures. Conversely, once one ascribes reality to the quantum state of a system, then one is confronted with the problem of explaining the collapse of the quantum state, which seems to contradict the dynamical laws of quantum mechanics (the Schrödinger equation). I propose to call this position, that takes at face value the (collapsed) states used to describe quantum mechanical systems, and interprets them as describing real properties of the system, the *naive realist* position in quantum mechanics (this term is by no means meant to be disparaging). However, there are other realist positions. The paradigm examples are Bohm theory and the modal interpretation. In both theories, only the state of the whole universe is real, in a sense, while properties of systems other than the universe are hardly at all related to the (collapsed) quantum states that are usually

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assigned to the systems. The main difference between Bohm theory and the modal interpretation, however, is that the description of the 'real state' of a system uses the *configuration space* of the system in the case of Bohm theory, and the *Hilbert space* of the system in the modal interpretation. The aim of this paper is to analyze how the modal interpretation, at least in some of its versions, fares with respect to the Kochen–Specker theorem, which, as is well known, poses restrictions on the ascription of truth values to propositions (projections) in the Hilbert space of a quantum mechanical system.

In this paper, I shall consider the versions of the modal interpretation by Kochen (1985), Healey (1989), and Dieks (1989). However, the approach taken here and the kind of questions addressed have much in common with work done by Bub in the context of his own version of the modal interpretation, for example, in Bub (1995). In Section 2, I shall briefly present the Kochen–Specker theorem and its framework. Then I shall introduce the rules of the modal interpretation (Section 3). I shall then motivate and derive a Kochen–Specker theorem for the modal interpretation (Sections 4 and 5). A brief discussion concludes the paper (Section 6).

2. PARTIAL BOOLEAN ALGEBRAS

The Kochen–Specker theorem is formulated in terms of partial Boolean algebras. A partial Boolean algebra, for short PBA [Kochen and Specker (1967) and references therein, all reprinted in Hooker (1975)], is a structure $(A;\ \heartsuit;\ \wedge, \vee, {}^{\perp}, 0, 1)$, where A is a set with two distinguished elements 0 and 1, $\heartsuit$ is a binary relation on A, called *comeasurability*, ${}^{\perp}$ is a unary operation on A (the *negation*), and $\wedge$ and $\vee$ are two partial binary operations on A, defined only for pairs of comeasurable elements (the *logical connectives*). Apart from the partial definability of the logical connectives, a PBA satisfies the usual axioms of a Boolean algebra, and every Boolean algebra is a PBA. The set of all subspaces of the Hilbert space of a quantum mechanical system forms a PBA, comeasurability being defined as commutativity of the corresponding projections, and the logical connectives being defined as the lattice-theoretic supremum and infimum, restricted to pairs of comeasurable propositions.

An *interpretation* of a PBA is defined as a pair

$$(B, \varphi) \tag{1}$$

where B is a partial Boolean (PB) subalgebra of the given PBA, and φ is a PBA-homomorphism of B onto the (partial) Boolean algebra $\{0, 1\}$. Since every Boolean algebra is (P)BA-homomorphic to $\{0, 1\}$, any embedding (PBA-homomorphism) of a PB subalgebra B into a Boolean algebra will induce interpretations of the PBA. Special cases are interpretations of the form

$$(C, \varphi) \tag{2}$$

where C is now a *Boolean* (or a maximal Boolean) subalgebra of the PBA.

Specker (1960) has emphasized that PBAs arise in the analysis of logics of propositions that are not simultaneously decidable. As we know from Bohr, quantum mechanical propositions that are not comeasurable in Kochen and Specker's sense are indeed not simultaneously decidable, because they require mutually incompatible experimental arrangements in order to be tested. In particular, no experiment can tell that two atomic propositions that are incomeasurable are true. However, there are interpretations of a quantum mechanical PBA in which incomeasurable atomic propositions are, indeed, simultaneously true! These can be constructed just by 'pasting together' two appropriate interpretations of the form (2). This situation is surprising, and prompts the question whether one could go on indefinitely pasting together interpretations, and obtain truth values for *all* quantum mechanical propositions.

The Kochen–Specker theorem (Kochen and Specker, 1967) answers this question in the negative. Kochen and Specker take a three-dimensional Hilbert space (the general case of dimension higher than three follows trivially), and they choose a finite sequence $(C_i)_{i=1,\dots,N}$ of maximal Boolean subalgebras (orthonormal bases), each element in the sequence being generated by a rotation about one of the vectors of the basis corresponding to the previous maximal Boolean subalgebra, so that any two successive elements have one proposition in common (the same holds for the first and the last elements in the sequence). The sequence is constructed in such a way that the union of the C_i is not embeddable in a Boolean algebra: whatever the choice of the φ_i, the truth valuations φ_i have no common extension to the union of all the maximal Boolean subalgebras C_i. This shows that the PBA of propositions is not embeddable in a Boolean algebra, because it contains a *finite* subalgebra that is not. Notice, however, that the general question of which PB subalgebras are embeddable in Boolean algebras is left open. And this is why we shall have to address a similar problem again in the context of the modal interpretation.

3. THE MODAL INTERPRETATION

I shall now spell out the basic rules of the modal interpretation, in the versions that are based on the spectral resolution of the reduced state of a system, or the biorthogonal decomposition theorem. I shall be concerned in particular with Healey's and Dieks' versions and with their further developments (I regard these as applying simultaneously to both versions) due to Vermaas and Dieks (1995) and Clifton (1995a). As I mentioned before, in the modal interpretation one assigns properties to quantum mechanical systems that correspond to projections in the Hilbert space of the system, but in a quite unusual way. Healey and Dieks ultimately disagree on *which* systems are to be ascribed properties, but they agree as to the rules by which these properties are ascribed.

First of all, no collapse of the state is assumed in the modal interpretation, so the usual, collapsed quantum state of a system plays no role in the determination of the properties of the system (it is only an effective state, as it were). One has to look at the 'true,' uncollapsed state of the system, that is, its *reduced* state ρ, obtained by partial tracing over the environment of the system. The usual way of ascribing properties to the system would be to take the projections with *dispersion-free values* as the definite properties of the system. That is, one defines an interpretation of the PBA of propositions that has the form

$$(B_\rho, \varphi_\rho) \tag{3}$$

where ρ is the reduced state of the system, B_ρ is the PB subalgebra generated by all projections with dispersion-free values in ρ, and φ_ρ is defined as the restriction of ρ to B_ρ. In the modal interpretation, one postulates that the system possesses additional properties with certain probabilities.[2] The rules use the unique spectral resolution of ρ:

$$\rho = \sum_i c_i P_i \tag{4}$$

They are as follows.

PROP. At any instant, the set of definite properties of a system with Hilbert space $\mathcal{H}$ in a state ρ is the PBA, call it A_ρ, generated by the projections with a dispersion-free value in ρ and by the projections P_i in the spectral resolution of ρ.

The possible truth-value assignments on A_ρ have the form φ_ρ^i, where $\varphi_\rho^i = \varphi_\rho$ on B_ρ, and $\varphi_\rho^i(P_j) = \delta_{ij}$.

PROB. The probability for the actual truth-value assignment on A_ρ to be φ_ρ^i is equal to

$$\mathrm{Tr}(\rho P_i) \tag{5}$$

Thus, A_ρ, the PBA of definite properties, is a proper extension of B_ρ, and a probability measure is given over the possible interpretations (A_ρ, φ_ρ^i). Comparison with the Born rule shows that the probabilities with

[2]The rules of the modal interpretation as originally formulated did not explicitly include the projections with dispersion-free values among the definite properties. It was Clifton (1995a), following Arntzenius (1990), who argued that these ought to be included in the set of definite properties, and who suggested modifying the rules accordingly.

which definite properties are actually true coincide with the probabilities for results of ideal measurements in standard quantum mechanics. If ρ is pure, the definite properties turn out to be *only* the projections with dispersion-free values in ρ, and the probabilities collapse to 0 or 1.

Healey and Dieks also agree as to the simultaneous ascription of properties when the systems that are to be ascribed properties are subsystems given by a factorization of a larger system. Take a system $\mathcal{H}$ in a state ρ, that is factorized as $\mathcal{H} = \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_N$. In this case, PROP is applied simultaneously to all subsystems in the factorization. The properties thus defined have then a joint probability distribution, fixed by the following condition.

CORR. The joint probability for the subsystems $\mathcal{H}_1, \ldots, \mathcal{H}_N$ of $\mathcal{H}$ to actually possess (by PROP) the properties $P^1_{i_1}, \ldots, P^N_{i_N}$, respectively, is equal to

$$\mathrm{Tr}(\rho P^1_{i_1} \otimes \ldots \otimes P^N_{i_N}) \tag{6}$$

Again, comparison with the Born rule shows that the joint probabilities for possessed values coincide with the quantum mechanical predictions for measurement outcomes upon ideal joint measurements. Some results about correlations between properties of a system and properties of its subsystems have recently been derived by Vermaas (1995).

However, the versions of Healey and Dieks diverge at the point of deciding *which* factorizations are allowed as defining subsystems to which properties are ascribed. Healey argues that certain factorizations are *physically preferred*, and ascribes properties *only* to the systems defined by those factorizations. Dieks instead considers *arbitrary* factorizations as defining systems that are ascribed properties. Thus Dieks ascribes properties to *every* quantum mechanical system, that is, to any subsystem appearing in any possible factorization of the Hilbert space of the universe. However, he has not yet formulated an explicit rule for correlations between properties of subsystems belonging to different factorizations. The Kochen–Specker theorem we are going to derive is, in fact, a constraint on any such rule for correlations between properties associated with different factorizations, and it suggests that these properties should, indeed, be uncorrelated.

Before continuing, let me mention that the above rules PROP, PROB, and CORR have an alternative formulation for the special case of a system $\mathcal{H}$ that is factorized into two subsystems and is in a pure state. If $|\Psi\rangle \in \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, then $|\Psi\rangle$ has a Schmidt (or biorthogonal) decomposition, that is, there are orthonormal bases $\{|\varphi_i\rangle\}$ in $\mathcal{H}_1$ and $\{|\psi_j\rangle\}$ in $\mathcal{H}_2$ such that

$$|\Psi\rangle = \sum_i c_i |\varphi_i\rangle \otimes |\psi_i\rangle \tag{7}$$

If the $|c_i|$ are nondegenerate, the choice of the bases is unique (up to phase factors). The alternative formulation of the rules now runs as follows.

SCHMIDT. If a system $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ is in a pure state $|\Psi\rangle$ with biorthogonal decomposition

$$|\Psi\rangle = \sum_i c_i |\varphi_i\rangle \otimes |\psi_i\rangle$$

then the system $\mathcal{H}$ possesses with probability 1 the properties given by the projections with the vector $|\Psi\rangle$ in their range, as well as with probability 0 all properties orthogonal to these. If the biorthogonal decomposition is unique (nondegenerate), the systems $\mathcal{H}_1$ and $\mathcal{H}_2$ possess the properties generated by the projections in $\mathcal{H}_1$ and $\mathcal{H}_2$ that have dispersion-free values in the respective reduced states (which are possessed with probabilities equal to 0 and 1) and by the projections onto the $|\varphi_i\rangle$ and $|\psi_j\rangle$ (which are possessed with joint probabilities equal to $|c_i|^2\delta_{ij}$). If the biorthogonal decomposition is degenerate, the additional generators of the properties of the subsystems are the corresponding multidimensional projections.

4. POSING THE PROBLEM

The standard example of an interpretation of the PBA of propositions of a quantum mechanical system is the pair (B_ρ, φ_ρ) described above (3), where ρ is the reduced state of the system. Interpretations that arise in this manner have a nice consistency property that, like the related concept introduced by Healey (1989) and analyzed by Clifton (1995b)[3], we shall call *property composition.*

Take a system $\mathcal{H}$ in a state ρ, that is factorized as $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. The interpretations of the form (3) defined on $\mathcal{H}_1$ and $\mathcal{H}_2$ are

$$(B_{\rho_1}, \varphi_{\rho_1}) \qquad \text{and} \qquad (B_{\rho_2}, \varphi_{\rho_2}) \tag{8}$$

where ρ_1 and ρ_2 are the reduced states of the subsystems $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. B_{ρ_1} and B_{ρ_2} are PB subalgebras of the PBAs of propositions of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, but since these are factor spaces in $\mathcal{H}$, one can consider B_{ρ_1} and B_{ρ_2} also as PB subalgebras of elements of the form $P \otimes \mathbf{1}$ and $\mathbf{1} \otimes Q$ *in the composite space* $\mathcal{H}$. So the interpretations (8) define also interpretations of the PBA of $\mathcal{H}$. But now, because of the definition of the

[3] Consider a quantum mechanical system with Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. Suppose that the actually possessed properties of the two subsystems are given respectively by the projections P in $\mathcal{H}_1$ and Q in $\mathcal{H}_2$. *Property composition* in the sense of Healey (1989) and Clifton (1995b) is the ascription of the *composite* property $P \otimes Q$ to the *total* system $\mathcal{H}$—in particular, the ascription of a property $P \otimes \mathbf{1}$ to the total system whenever the corresponding subsystem possesses a property P.

reduced states ρ_1 and ρ_2, it is evident that the interpretations of the PBA of $\mathcal{H}$ thus defined *have a common extension* that is, in fact, a subinterpretation of (B_ρ, φ_ρ). Furthermore, this holds irrespective of the chosen factorization. That is, given a system $\mathcal{H}$ in a state ρ, *all* interpretations of the form (3) defined on all subsystems of $\mathcal{H}$ with respect to any possible factorization (including the trivial factorization $\mathcal{H} = \mathbf{C} \otimes \mathcal{H}$) have a common extension, and this common extension is (B_ρ, φ_ρ). The interpretations of the form (3) can be 'pasted together' and the union of all B_ρ, defined in the large Hilbert space, is *embeddable in a Boolean algebra.*

The question we address in this paper is whether, when we define properties of systems using the A_ρ instead of the B_ρ, the corresponding PB subalgebras in the composite space $\mathcal{H}$ also are jointly embeddable in a Boolean algebra, or whether this is precluded by a result similar to the Kochen–Specker theorem. Interpretations for systems defined by the same factorization do have a common extension, because they define mutually comeasurable PB subalgebras in $\mathcal{H}$. Interpretations defined for systems and their subsystems (as in Healey's version) can probably also be pasted together regardless of the Kochen–Specker theorem, although this needs to be shown explicitly. We shall investigate whether PBAs of definite properties of systems defined, as in Dieks' modal interpretation, by *different* factorizations of a given system form a PBA that is embeddable in a Boolean algebra. We shall show that they cannot.

5. THE KOCHEN–SPECKER THEOREM

5.1. Factorizations of Hilbert Spaces

One can define a *factorization* of a Hilbert space $\mathcal{H}$ into a tensor product of two Hilbert spaces $\mathcal{H}_1 \otimes \mathcal{H}_2$ as an isomorphism between $\mathcal{H}$ and $\mathcal{H}_1 \otimes \mathcal{H}_2$, or better as an equivalence class of isomorphisms differing only by a basis transformation of the factor spaces onto themselves.

Definition 1. Let $\mathcal{H}$ be an $(n \times m)$-dimensional Hilbert space (where n and m could be infinite). An *$(n \times m)$-factorization* Φ of $\mathcal{H}$ is an equivalence class of isomorphisms

$$f\colon \mathcal{H} \to \mathcal{H}_1 \otimes \mathcal{H}_2$$

(where $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces of dimension n and m, respectively), defined by the following equivalence relation:

$$\tilde{f} \sim f :\Leftrightarrow \tilde{f} = V_1 \otimes V_2 f$$

with V_1: $\mathcal{H}_1 \to \mathcal{H}_1$ and V_2: $\mathcal{H}_2 \to \mathcal{H}_2$ both unitary.

Definition 2. A unitary transformation U: $\mathcal{H} \to \mathcal{H}$ is *factorizable* with respect to a factorization Φ iff for an $f \in \Phi$ (and hence for all $f \in \Phi$)

$$fU = U_1 \otimes U_2 f$$

with U_1: $\mathcal{H}_1 \to \mathcal{H}_1$ and U_2: $\mathcal{H}_2 \to \mathcal{H}_2$ both unitary.

From these two definitions one derives immediately the following lemma.

Lemma 3. Let f be an isomorphism from $\mathcal{H}$ onto $\mathcal{H}_1 \otimes \mathcal{H}_2$, and let U be a unitary transformation of $\mathcal{H}$ onto itself. Define a new isomorphism $\tilde{f} :=$ fU. Then

$$\tilde{f} \sim f \Leftrightarrow U \text{ is factorizable}$$

We can now formulate the biorthogonal decomposition theorem in the terminology of factorizations.

Theorem 4. (Schmidt). Let Φ be a factorization of a Hilbert space $\mathcal{H}$. Then for all $|\Psi\rangle \in \mathcal{H}$ there is an $f_S \in \Phi$ such that

$$|\Psi\rangle = \sum_i c_i f_S^{-1} |\varphi_i\rangle \otimes |\psi_i\rangle$$

where $\{|\varphi_i\rangle\}$ and $\{|\psi_i\rangle\}$ are two fixed orthonormal bases in $\mathcal{H}_1$ and $\mathcal{H}_2$. If the $|c_i|$ are nondegenerate, f_S is unique (apart from phase factors).

5.2. Proof of the Theorem

We now prove the Kochen–Specker theorem for the modal interpretation. It is enough to specialize to the case of a 9-dimensional Hilbert space and its (3×3)-dimensional factorizations; the general case then follows. We shall assume throughout that all biorthogonal decompositions are nondegenerate, and discuss this assumption later.

Take a $|\Psi\rangle \in \mathcal{H}$ and a factorization Φ. Then the biorthogonal decomposition of $|\Psi\rangle$ with respect to this factorization is

$$|\Psi\rangle = c_1 f_S^{-1} |\varphi_1\rangle \otimes |\psi_1\rangle + c_2 f_S^{-1} |\varphi_2\rangle \otimes |\psi_2\rangle + c_3 f_S^{-1} |\varphi_3\rangle \otimes |\psi_3\rangle \quad (9)$$

Comparison with SCHMIDT yields that the properties jointly possessed by the two subsystems (ignoring those with vanishing joint probability) correspond in the *composite* Hilbert space $\mathcal{H}$ to the projections onto the vectors

$$f_S^{-1} |\varphi_i\rangle \otimes |\psi_i\rangle \quad (10)$$

and these projections define a PB subalgebra (in this case a Boolean subalgebra) of the PBA of propositions of $\mathcal{H}$.

Now consider the 3-dimensional subspace $\mathcal{K}$ of $\mathcal{H}$ spanned by

$$\{f_S^{-1}|\varphi_1\rangle \otimes |\psi_1\rangle, f_S^{-1}|\varphi_2\rangle \otimes |\psi_2\rangle, f_S^{-1}|\varphi_3\rangle \otimes |\psi_3\rangle\} \tag{11}$$

In $\mathcal{K}$, the definite properties defined by the factorization Φ in $\mathcal{H}$ form a maximal Boolean subalgebra. We shall reproduce in this subspace the Kochen–Specker construction sketched above.

Take a unitary transformation U: $\mathcal{K} \to \mathcal{K}$, with $U \neq \mathbf{1}_{\mathcal{K}}$, and extend it trivially to a unitary transformation $\tilde{U}$: $\mathcal{H} \to \mathcal{H}$, that is,

$$\tilde{U}f_S^{-1}|\varphi_i\rangle \otimes |\psi_j\rangle := \begin{cases} Uf_S^{-1}|\varphi_i\rangle \otimes |\psi_i\rangle, & i = j \\ f_S^{-1}|\varphi_i\rangle \otimes |\psi_j\rangle, & i \neq j \end{cases} \tag{12}$$

We now prove the following lemma.

Lemma 5. Let $\tilde{U}$ be defined as above. Then $\tilde{U}$ is not factorizable.

Proof. Assume that $\tilde{U}$ is factorizable with respect to Φ, that is, assume $f_S\tilde{U}f_S^{-1} = U_1 \otimes U_2$. Then for all $i \neq j$ we have, by (12),

$$\begin{aligned} f_S\tilde{U}f_S^{-1}|\varphi_i\rangle \otimes |\psi_j\rangle &= U_1|\varphi_i\rangle \otimes U_2|\psi_j\rangle \\ &= |\varphi_i\rangle \otimes |\psi_j\rangle \end{aligned}$$

Thus

$$U_1 = \mathbf{1}_{\mathcal{H}_1}, \qquad U_2 = \mathbf{1}_{\mathcal{H}_2}, \qquad \tilde{U} = \mathbf{1}_{\mathcal{H}}, \qquad U = \mathbf{1}_{\mathcal{K}}$$

thus contradicting the assumptions. QED.

From Lemma 3 it follows that

$$\tilde{\Phi} := \{\tilde{f}|\tilde{f} = f\tilde{U}, f \in \Phi\} \tag{13}$$

is a factorization distinct from Φ. Thus, by Theorem 4 and SCHMIDT, we obtain that the properties jointly possessed by the systems defined by this new factorization correspond to a Boolean subalgebra in $\mathcal{H}$ that is generated by the projections onto the vectors

$$\tilde{f}_S^{-1}|\varphi_i\rangle \otimes |\psi_i\rangle \tag{14}$$

(recall we assume all biorthogonal decompositions to be nondegenerate). But now, these projections form a maximal Boolean subalgebra in $\mathcal{K}$ bearing a specific relation to the one in (10), defined by the factorization Φ:

Lemma 6. Let f_S and $\tilde{f}_S$ be defined as before. Then

$$\tilde{f}_S = f_S\tilde{U}$$

Proof. Define $\tilde{f} := f_S\tilde{U}$. We show that $\tilde{f} = \tilde{f}_S$. We have for all i, j

$$\tilde{f}^{-1}|\varphi_i\rangle \otimes |\psi_j\rangle = \tilde{U}^{-1}f_S^{-1}|\varphi_i\rangle \otimes |\psi_j\rangle$$

and since $\tilde{U}$ extends U: $\mathcal{H} \to \mathcal{H}$, the $\tilde{f}^{-1}|\varphi_i\rangle \otimes |\psi_i\rangle$ are linear combinations of the $f_S^{-1}|\varphi_i\rangle \otimes |\psi_i\rangle$ alone, and vice versa. This means that

$$\begin{aligned}|\Psi\rangle &= c_1f_S^{-1}|\varphi_1\rangle \otimes |\psi_1\rangle + c_2f_S^{-1}|\varphi_2\rangle \otimes |\psi_2\rangle + c_3f_S^{-1}|\varphi_3\rangle \otimes |\psi_3\rangle \\ &= \tilde{c}_1\tilde{f}^{-1}|\varphi_1\rangle \otimes |\psi_1\rangle + \tilde{c}_2\tilde{f}^{-1}|\varphi_2\rangle \otimes |\psi_2\rangle + \tilde{c}_3\tilde{f}^{-1}|\varphi_3\rangle \otimes |\psi_3\rangle\end{aligned}$$

for some coefficients $\tilde{c}_i$. So, indeed, $\tilde{f}_S = \tilde{f}$ and $\tilde{f}_S = f_S\tilde{U}$. QED.

But now we have shown that the definite properties given by $\tilde{\Phi}$ are generated by the unitary transformation U^{-1} in $\mathcal{H}$ from the definite properties given by Φ. And this is what we need to generate a Kochen–Specker contradiction! In fact, we can now take the finite sequence of rotations constructed by Kochen and Specker and obtain a sequence of interpretations of the form (2) in the 3-dimensional space $\mathcal{H}$, each of which corresponds, by the above, to definite properties jointly possessed by the subsystems defined by some factorization of $\mathcal{H}$. By the Kochen–Specker theorem we know that the PB subalgebra generated by these maximal Boolean subalgebras does *not* admit a PB-homomorphism onto $\{0, 1\}$!

So we have proved the main theorem of this paper:

Theorem 7. The union of the maximal Boolean subalgebras defined according to (10) by a pure state $|\Psi\rangle$ and by the different factorizations Φ of a 9-dimensional Hilbert space $\mathcal{H}$ is not embeddable in a Boolean algebra.

We still have to discuss the case in which at some stage in our construction we encounter degeneracy of the coefficients in some biorthogonal decomposition. However, this will be the case only for very special choices of the initial factorization Φ. For each $|\Psi\rangle$ there is an initial factorization Φ such that the construction can be carried out. In this sense, our assumption about nondegeneracy of biorthogonal decompositions does not imply a loss of generality.

6. CONCLUSIONS

Where do we stand now? The Kochen–Specker result we have derived implies there are very deep differences between the two versions of the modal interpretation we have considered, Healey's and Dieks'. Indeed, it forces a choice between the two versions as between two horns of a dilemma. Healey's choice of systems to which he ascribes properties may well allow for a common extension to the whole universe, so that Healey can retain a view about how subsystems relate to the whole that is the same as in standard quantum mechanics and metaphysically familiar. He has, however, to justify the postulate of a preferred (N-, or infinite) factorization of the universe.

Dieks instead has to introduce a new kind of metaphysics, one in which the notion of a quantum mechanical system becomes primary, and one in which properties of systems belonging to different factorizations seem not to be correlated. Further consequences for Dieks' version of the modal interpretation are discussed in Bacciagaluppi (1995).

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REFERENCES

Arntzenius, F. (1990). Kochen's interpretation of quantum mechanics, in Philosophy of Science Association, A. Fine, M. Forbes, and L. Wessels, eds., *Proceedings of the Philosophy of Science Association 1990*, East Lansing, Michigan, Vol. 1, pp. 241–249.

Bacciagaluppi, G. (1995). Ph.D. thesis, in preparation.

Bub, J. (1995). Maximal structures of determinate propositions in quantum mechanics, *International Journal of Theoretical Physics*, this issue.

Clifton, R. (1995a). Independently motivating the Kochen–Dieks modal interpretation of quantum mechanics, *British Journal for the Philosophy of Science*, **46**, 33–57.

Clifton, R. (1995b). Why modal interpretations of quantum mechanics must abandon "classical" reasoning about the values of observables, *International Journal of Theoretical Physics*, this issue.

Dieks, D. (1989). Resolution of the measurement problem through decoherence of the quantum state, *Physics Letters A*, **142**, 439–446.

Healey, R. (1989). *The Philosophy of Quantum Mechanics: An Interactive Interpretation*, Cambridge University Press, Cambridge.

Hooker, C. A., ed. (1975). *The Logico-Algebraic Approach to Quantum Mechanics. Volume I: Historical Evolution*, Reidel, Dordrecht.

Kochen, S. (1985). A new interpretation of quantum mechanics, in *Symposium on the Foundations of Modern Physics*, P. Lahti and P. Mittelstaedt, eds., World Scientific, Singapore, pp. 151–169.

Kochen, S., and Specker, E. P. (1967). On the problem of hidden variables in quantum mechanics, *Journal of Mathematics and Mechanics*, **17,** 59–87 [Reprinted in Hooker (1975), pp. 293–328].

Specker, E. P. (1960). Die Logik nicht gleichzeitig entscheidbarer Aussagen, *Dialectica*, **14,** 239–246 [Transl., The logic of propositions which are not simultaneously decidable, in Hooker (1975), pp. 135–140].

Vermaas, P. E. (1995). Unique Transition Probabilities in the Modal Interpretation, preprint.

Vermaas, P. E., and Dieks, D. (1994). The modal interpretation of quantum mechanics and its generalization to density operators, *Foundations of Physics*, **25**, 145–158.